OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2611**

Mechanics 5

Monday **23 JUNE 2003** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8$ m s^{-2} unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

[Turn over

1 *Variable Forces*

A train of total mass m is pulled by an engine which generates constant power P. After starting from rest, the train is subject to a resistance kv^2, where v is the speed of the train at time t and k is a constant. The distance travelled is denoted by x.

(i) Write down a differential equation giving the acceleration $v\frac{dv}{dx}$ of the train on a straight level track in terms of P, m, k, and v.

Deduce an expression in terms of P and k for V_0, the terminal speed of the train. [5]

(ii) By solving the differential equation, show that $v^3 = V_0^3\left(1-e^{-3kx/m}\right)$. [8]

When the train reaches half its terminal speed, it has travelled a distance H and the power is turned off.

(iii) Show that, when the train has travelled a further distance H, the speed has dropped to approximately $0.48V_0$. [7]

2 *Relative Motion*

The position vectors $\mathbf{r}_E$ of the Earth and $\mathbf{r}_M$ of Mars relative to an origin at the Sun are modelled by

$$\mathbf{r}_E = R(\mathbf{i}\cos\alpha t + \mathbf{j}\sin\alpha t),$$

$$\mathbf{r}_M = S(\mathbf{i}\cos\beta t + \mathbf{j}\sin\beta t),$$

where α, β, R and S are constants and t is measured in years for $0 \leqslant t < 2$.

It is known that Mars takes two years to orbit the Sun and is 1.5 times as far from the Sun as the Earth is.

(i) Explain why $\alpha = 2\pi$. Write down a corresponding value for β and express S in terms of R. [3]

(ii) Write down in terms of R and t the position vector of Mars relative to the Earth. Show that the distance between the two planets at time t is $R(3.25-3\cos\pi t)^{\frac{1}{2}}$. Hence or otherwise find the greatest and least separations of the planets in terms of R. [7]

(iii) Find the relative velocity vector of the two planets.

By considering the two components separately, show that the relative velocity of the two planets is never zero. Explain carefully why the speed of separation of the planets is not always the same as the relative speed of the planets. [10]

3 *Motion described in Polar Coordinates*

The path of a particle is described by the equation $r = 3 + 2\cos\theta$, where (r, θ) are plane polar coordinates. The unit vectors in the radial and transverse directions are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$.

At time t, $\theta = \omega t$ where ω is a constant and $t \geqslant 0$.

(i) Using the result $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$, show that the velocity vector $\dot{\mathbf{r}}$ is given by

$$\dot{\mathbf{r}} = -(2\omega\sin\omega t)\hat{\mathbf{r}} + \omega(3+2\cos\omega t)\,\hat{\boldsymbol{\theta}}.$$

Using the result $\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$, find a similar result for $\ddot{\mathbf{r}}$. [6]

(ii) Deduce expressions for the speed and for the magnitude of the acceleration. Find the minimum values of these quantities and the times at which they occur. Show that at each of these times, the velocity and acceleration vectors are perpendicular. Find all the other times at which the velocity and acceleration vectors are perpendicular. [12]

(iii) Explain why it is *not* possible for the particle to describe the given path under the action of a central force only. [2]

Mark Scheme

Question 1

(i)

$mv\frac{dv}{dx} = \frac{P}{v} - kv^2$	M1A1A1	
Terminal speed when $\frac{dv}{dx} = 0 \therefore V_0 = \left(\frac{P}{k}\right)^{\frac{1}{3}}$	M1A1	[5]
(ii)		
$\int \frac{v^2 dv}{P - kv^3} = \int \frac{dx}{m}$,	M1A1	
$\ln(P - kv^3) = \frac{-3kx}{m} + C$	M1A1	
Using boundary conditions gives $C = \ln P$	M1A1	
$v^3 = V_0^3\left(1 - e^{\frac{-3kx}{m}}\right)$	M1E1, **ag**	[8]
(iii)		
$\exp\left(\frac{-3kH}{m}\right) = \frac{7}{8}$	M1A1	
$\int \frac{dv}{v} = \int \frac{-kdx}{m}, \therefore \ln v = \frac{-kx}{m} + D,$ where $D = \ln\frac{V_0}{2}$,	M1,A1,M1	
$v = \frac{V_0}{2}\left(\frac{7}{8}\right)^{\frac{1}{3}} = 0.48V_0$	M1E1, **ag**	[7]

Question 2

(i)		
Explain that $\alpha = 2\pi$.	B1	
State $\beta = \pi \quad S = 1.5R$	B1, B1	[3]
(ii)		
${}_{M}\mathbf{r}_{E} = R\mathbf{i}(1.5\cos\pi t - \cos 2\pi t) + R\mathbf{j}(1.5\sin\pi t - \sin 2\pi t)$	M1A1	
$\lvert {}_{M}\mathbf{r}_{E} \rvert^2 = R^2(1.5\cos\pi t \ldots)^2 + R^2(1.5\sin\pi t \ldots)^2$	M1A1	
$\lvert {}_{M}\mathbf{r}_{E} \rvert = R(3.25 - 3\cos\pi t)^{\frac{1}{2}}$	A1, **ag**	
max distance $= R(6.25)^{\frac{1}{2}} = 2.5R$	B1	
min distance $= R(0.25)^{\frac{1}{2}} = 0.5R$	B1	[7]
(iii)		
${}_{M}\dot{\mathbf{r}}_{E} = \pi R\mathbf{i}(-1.5\sin\pi t + 2\sin 2\pi t) + \pi R\mathbf{j}(1.5\cos\pi t - 2\cos 2\pi t)$	M1A1	
${}_{M}\dot{\mathbf{r}}_{E} = 0 \Rightarrow 1.5\sin\pi t = 2\sin 2\pi t$ and $1.5\cos\pi t = 2\cos 2\pi t$	M1A1	
But $1.5^2 \neq 2^2 \Rightarrow$ not possible	M1A1	
Speed of separation is the radial speed only.	M1A1	
The relative speed includes both radial and transverse cmpts	B1	
Equal only when transverse component is zero	B1	[10]

Question 3

(i)

$\dot{r} = -2\omega\sin\omega t, \quad \ddot{r} = -2\omega^2\cos\omega t, \quad \dot{\theta} = \omega, \quad (\ddot{\theta} = 0)$.	B1,B1,B1	
$\dot{\mathbf{r}} = -2\omega\sin\omega t\,\hat{\mathbf{r}} + \omega(3+2\cos\omega t)\hat{\theta}$	B1, ag	
$\ddot{\mathbf{r}} = -\omega^2(4\cos\omega t + 3)\hat{\mathbf{r}} - 4\omega^2\sin\omega t\,\hat{\theta}$	M1A1	[6]
(ii)		
$\lvert\dot{\mathbf{r}}\rvert = \omega(13+12\cos\omega t)^{\frac{1}{2}}$	M1A1	
minimum value $=\omega$, and $\omega t = (2n+1)\pi$	B1, B1(cao)	
$\lvert\ddot{\mathbf{r}}\rvert = \omega^2(25+24\cos\omega t)^{\frac{1}{2}}$	M1A1	
$\Rightarrow$ minimum value $= \omega^2$, and $\omega t = (2n+1)\pi$	B1, B1(cao)	
At $\omega t = (2n+1)\pi$, $\dot{\mathbf{r}} = \omega\hat{\theta}$, $\ddot{\mathbf{r}} = \omega^2\hat{\mathbf{r}}$, $\Rightarrow \dot{\mathbf{r}}.\ddot{\mathbf{r}} = 0$	M1A1	
$\dot{\mathbf{r}}.\ddot{\mathbf{r}} = \omega^3(8\sin\omega t\cos\omega t + 6\sin\omega t - 12\sin\omega t - 8\sin\omega t\cos\omega t)$		
$\therefore\ \dot{\mathbf{r}}.\ddot{\mathbf{r}} = -6\omega^3\sin\omega t = 0$, when $\omega t = n\pi$ for integer n.		
Hence other times are at $\dfrac{2n\pi}{\omega}$	M1A1(cao)	[12]
(iii)		
For central force motion, $\ddot{\mathbf{r}}$ and hence the force must be radial.	M1A1	[2]

Question 4

(i)

$I = \int \left(\frac{2\pi r dr}{\pi a^2}\right) m r^2 = \frac{ma^2}{2}$	M1A1, M1A1	
M of I about any diam $= I_D$		
Perp axis thm $\Rightarrow I = 2I_D \Rightarrow I_D = \frac{ma^2}{4}$	M1A1	[6]
(ii)		
$\frac{ma^2}{2}\ddot{\theta} = -K\dot{\theta}$	M1A1	
$\dot{\theta} = \frac{-2K\theta}{ma^2} + \omega = 0$ when $\theta = \frac{ma^2\omega}{2K}$	M1A1, A1, **ag**	[5]
(iii)		
Use of symmetry (or moments) $Mb = 2m_1 c$, (c of g)	M1A1	
Explaining that $Mb^2 = 2m_1 a^2$ (m of i)	M1A1,**ag**	
Hence $a^2 = \frac{Mb^2}{2m_1} = \frac{b^2}{2}\frac{2c}{b} = bc$	M1A1, A1	
But $b < a$, and $c < a$ $\therefore bc < a^2$ and hence impossible	M1A1 **ag**	[9]

Examiner’s Report

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General Comments

Just as in 2002, there were 68 candidates for this paper and once again there was a great deal of work of high quality. Some candidates had difficulty with some parts of the questions but most more than made up for it elsewhere and there were very few really weak answers. As has been noted in the past, some candidates struggle with algebraic manipulation but this was pleasingly not so much of a problem this year.

Comments on Individual Questions

Q.1 Variable forces

The candidates had little trouble with this question about a train pulled by a variable force under a variable resistance. Most knew how to relate power and force and soon completed the first two parts of the question. Once the differential equation had been solved, the requirement was to solve the equation again but under no tractive force. Some omitted to do this and tried to establish the final given result from the first part of the question. Those who solved the equation again usually scored full marks.

(i) $V_0 = \left(\frac{P}{k}\right)^{\frac{1}{3}}$.

Q.2 Relative motion

The only part of this vector question on the relative motion of Mars and the Earth that the candidates found difficult was the final request to explain carefully the difference between the speed of separation of the two planets and their relative velocity. Many knew the answer by instinct but found it very difficult, if not impossible, to explain themselves. Quite a few said that it was because the speed of separation has a sign (plus or minus) whereas the relative velocity does not. The answer expected was that the speed of separation is the component of the relative velocity along the line of centres of the planets.

In part (ii), many candidates made extremely hard work of finding the maximum and minimum values of an expression like $(A - \cos x)^{\frac{1}{3}}$ choosing to use calculus, including distinguishing the two values with the second derivative!

(i) $\beta = \pi$ and $S = 1.5R$.; (ii) $2.5R$ and $0.5R$ respectively.

Q.3 Motion described in polar coordinates

The candidates were very unsure of themselves when establishing results in part (i) . In part (ii) they made heavy weather of finding maximum and minimum values of simple trigonometric functions, although they did get there in the end.

(ii) Speed and acceleration extrema occur at times $t = \frac{(2n+1)\pi}{\omega}$.

Velocity and acceleration vectors are also perpendicular at times $t = \frac{2n\pi}{\omega}$.

Q.4 Rotation of a rigid body

The establishment of the moments of inertia in part (i) was exemplary. It was surprising how many candidates had difficulty in stating and then solving the equation of motion for the rotation in part (ii).

A significant number attempted to use an energy equation, presumably by analogy with linear motion. This is fraught with difficulty. Worse though were those who used linear constant acceleration formulae. Answers to part (iii) were actually quite good whatever happened earlier in the question.

(iii) $a^2 = bc$.